ISRAEL JOURNAL OF MATHEMATICS, Vol. 44, No. 3, 1983

STRONG LIFTINGS WITH APPLICATION TO MEASURABLE CROSS SECTIONS IN LOCALLY COMPACT GROUPS

BY
JOSEPH KUPKA

ABSTRACT

There are two principal theorems. The adjustment theorem asserts that a lifting may be changed on a set of measure zero so as to become slightly stronger. In conjunction with the standard lifting theorem, it yields generalizations (with shorter proofs) of a number of known results in the theory of strong liftings. It also inspires a characterization of strong liftings, when the measure is regular, by the fact that they induce upon every open set an artificial "closure" of that set which differs from it by a set of measure zero. The projection theorem asserts that, in the presence of a strict disintegration, a strong lifting may be transferred or "projected" from one topological measure space onto another. In conjunction with Losert's example, it yields regular Borel measures, carried on compact Hausdorff spaces of arbitrarily large weight, which everywhere fail to have the strong lifting property. It also provides the final link needed to obtain, with no separability assumptions, a measurable cross section (or right inverse) for the canonical map $\psi : G \rightarrow G/H$, where G is an arbitrary locally compact group, and where H is an arbitrary closed subgroup of G.

1. Introduction

Beginning with the paper of von Neumann [23] (cf. [24, theorem 18, p. 372]), it has been customary to obtain a measure theoretic lifting in a step-by-step constructive manner, using Zorn's lemma or transfinite induction ([22], [14], [16, chapter IV], [28], [9], [7], etc.). An alternate theme which is at least implicit in the literature (see, for example, [3, proposition 5, p. 407]) is to obtain a lifting by transforming a pre-existing lifting in some fashion. It is this latter theme, as it applies to the construction of strong liftings, which we shall explore more fully in the present paper. There are two principal results. The adjustment theorem (Theorem 2.1) asserts that a given lifting may be "adjusted" on a set of measure zero to become strong with respect to that set. Simple in itself, it yields

Received March 9, 1982

generalizations (with short proofs) of a number of known results in the theory of strong liftings, some of whose constructive proofs are considerably longer; it also leads to a characterization (in Theorem 3.4) of the strong lifting property for regular measures which is obtained by "globalizing" the adjustment procedure. The projection theorem (Theorem 2.7) presents conditions under which a strong lifting may be "projected" from one space onto another. It yields (in Example 3.2) a slight extension of Losert's counterexample to the strong lifting conjecture [20] and leads (in Section 4) to a general existence theorem for measurable cross sections in locally compact groups.

All measures mentioned in this paper will be nonnegative, countably additive, not necessarily finite, and, unless specified otherwise, nonzero. A *measure space* is a triple (X,Σ,μ), where X is a nonempty set, where Σ is a σ-algebra of subsets of X, and where μ is a measure with domain Σ. We shall say that X constitutes the *direct sum mod* μ of a family $\{X_\alpha\}$ of pairwise disjoint subsets of X if, for all $A \subseteq X$, the following conditions are satisfied:

(1.1) $\qquad A \in \Sigma$ if and only if $A \cap X_\alpha \in \Sigma$ for each index α;

(1.2) $\qquad$ if $A \in \Sigma$, then $\mu(A)=\Sigma_\alpha \mu(A \cap X_\alpha)$, where Σ_α denotes the supremum (in the extended real numbers) over the collection of finite sums.

If $\mathcal{T} \subseteq \Sigma$ is a topology, then the quadruple $(X,\Sigma,\mu,\mathcal{T})$ will be referred to as a *topological measure space*. A topological measure space is *complete* if it is complete as a measure space; if $\mathcal{S} \subseteq \Sigma$ is a σ-ring, the completion of $\mathcal{S}$ with respect to $\mu|_{\mathcal{S}}$ will be denoted by $\mathcal{S}_\mu$. A set $A \in \Sigma$ will be called a *carrier* of μ, and μ will be said to be *carried* on A if, for all $U \in \mathcal{T}$, we have $\mu(A \cap U)>0$ whenever $A \cap U \neq \varnothing$. We declare that μ is *regular* if, for all $A \in \Sigma$, we have

$$\mu(A)=\sup\{\mu(K): K \text{ closed};\ K \subseteq A\}. \tag{1.3}$$

When X is a locally compact Hausdorff space, we define $\mathcal{B}_\delta(X)$ (resp. $\mathcal{B}_\sigma(X)$) to be the δ-ring (resp. σ-ring) generated by the compact subsets of X, and we define $\mathcal{B}_l(X)$, the locally measurable sets, to be the σ-algebra of all sets A such that $A \cap E \in \mathcal{B}_\sigma(X)$ for all $E \in \mathcal{B}_\sigma(X)$. If a (standard) regular Borel measure μ has been defined initially on $\mathcal{B}_\delta(X)$ or on $\mathcal{B}_\sigma(X)$, we assume that it has been extended to $\mathcal{B}_l(X)$ via the formula

$$\mu(E)=\sup\{\mu(A): A \in \mathcal{B}_\delta(X);\ A \subseteq E\}.$$

Clearly this extended measure μ will be regular in the sense of (1.3). Any lifting for μ will be assumed to have domain $\Sigma=\mathcal{B}_l(X)_\mu$.

Set notation will be standard. In particular, we shall write A^c for $X \setminus A$ and $A \triangle B$ for $(A \setminus B) \cup (B \setminus A)$. *Whether or not A and B are measurable*, the relation $A \subseteq_\mu B$ will mean that the set $A \setminus B$ is μ-null. If $A \subseteq_\mu B$ and $B \subseteq_\mu A$, then we shall write $A \approx_\mu B$. If $E \in \Sigma$, and if $\mathscr{S} \subseteq \Sigma$, we let $\mathscr{S}|_E = \{A \cap E : A \in \mathscr{S}\}$, and we let $\mu|_E$ denote the restriction of μ to $\Sigma|_E$.

A *lifting for μ on* (X, Σ) is a function $\rho : \Sigma \to \Sigma$ with the following properties, which hold for all $A, B \in \Sigma$:

$$\rho(A) \approx_\mu A; \tag{1.4}$$

$$\text{if } A \approx_\mu B, \text{ then } \rho(A) = \rho(B); \tag{1.5}$$

$$\rho(\varnothing) = \varnothing \quad \text{and} \quad \rho(X) = X; \tag{1.6}$$

$$\rho(A \cap B) = \rho(A) \cap \rho(B); \tag{1.7}$$

$$\rho(A \cup B) = \rho(A) \cup \rho(B). \tag{1.8}$$

If ρ satisfies (1.4)–(1.7), then ρ is called a *density* (for μ on (X, Σ)). If the measure space is topological with topology $\mathscr{T}$, then a density or lifting ρ will be called *strong* (with respect to $\mathscr{T}$) if

$$\rho(U) \supseteq U \qquad \text{for all } U \in \mathscr{T}. \tag{1.9}$$

If such a lifting exists, then $(X, \Sigma, \mu, \mathscr{T})$ (or just μ) is said to have the *strong lifting property*.

Let (X, Σ, μ) and (Y, Υ, ν) be measure spaces, let $\mathscr{S} \subseteq \Sigma$ be a σ-ring, and let $\psi : X \to Y$ satisfy (at least) the condition: for all $y \in Y$, the inverse image $\psi^{-1}(\{y\})$ is a nonempty element of Σ. Then a family $\{\mu_y\}_{y \in Y}$ of measures on (X, Σ) will be said to constitute a *strict disintegration of μ with respect to ν induced on $(X, \mathscr{S})$ by ψ* (cf. [19, definition 2.1, p. 8]) if the following conditions are satisfied:

$$\mu_y(X) > 0 \quad \text{a.e. } (\nu); \tag{1.10}$$

$$\mu_y(\psi^{-1}(\{y\})^c) = 0 \quad \text{a.e. } (\nu); \tag{1.11}$$

$$\text{for all } A \in \mathscr{S}, \text{ the function } y \mapsto \mu_y(A) \text{ is } \nu\text{-measurable}, \tag{1.12}$$

and we have

$$\mu(A) = \int \mu_y(A) d\nu(y).$$

Strict disintegrations are known to exist mainly in the topological setting, where they are closely tied in with strong liftings (see [16, theorem 5, p. 150] and [12]).

In view of [19, example 5.2, p. 32], it seems unlikely that (strict) disintegrations exist when $\mathcal{S}$ is not contained in the σ-ring of σ-finite subsets of X. We shall be concerned with the following two examples, both of which are defined without reference to liftings.

1.13. EXAMPLE. Let X and Y be compact Hausdorff spaces, so that we may and shall define $\mathcal{B}(X)=\mathcal{B}_\delta(X)=\mathcal{B}_\sigma(X)=\mathcal{B}_l(X)$; and let μ and ν be regular Borel measures on X and Y, respectively. Then by [17, theorem 2.1, p. 113], the product measure $\mu\times\nu$ can be realized as a (unique) regular Borel measure, and by [17, theorem 4.5, p. 119], the formula

$$\nu_x(A)=\int \chi_A(x,y)d\nu(y)$$

(where $x\in X$ and $A\in\mathcal{B}(X\times Y)$) determines a strict disintegration of $\mu\times\nu$ with respect to μ induced on $(X\times Y,\mathcal{B}(X\times Y))$ by the canonical projection π of $X\times Y$ onto X. ■

1.14. EXAMPLE. Let G be a locally compact Hausdorff topological group, and let H be a closed subgroup of G. Let G/H denote the set of left cosets of G, which becomes a locally compact Hausdorff space when endowed with the quotient topology induced by the canonical map $\psi: G\to G/H$ [1, p. 39]. Where convenient we shall write $\dot{x}$ in place of $\psi(x)$. Now let μ and β be left Haar measures and let Δ and δ denote the modular functions on G and H, respectively. By [1, theorem 2, p. 56] there is a strictly positive continuous function ρ defined on G such that $\rho(xs)=\Delta(s)\delta(s)^{-1}\rho(x)$ for all $x\in G$ and $s\in H$; and in consequence one may determine a quasi-invariant regular Borel measure λ on G/H with respect to which there exists a strict disintegration of μ induced on $(G,\mathcal{B}_\sigma(G))$ by ψ. To describe this disintegration, it is convenient to refer to [26, proposition 10.1, p. 482], which implies that, for all $x\in G$ and $A\in\mathcal{B}_\sigma(G)$, the expression

$$\rho(x)^{-1}\int_H \frac{\Delta(s)}{\delta(s)}\chi_A(xs)d\beta(s)$$

is well defined and constant on the left cosets of H, so that it determines the value at A of a nonzero regular Borel measure $\mu_{\dot{x}}$ defined on G and carried on $xH=\psi^{-1}(\{\dot{x}\})$. Moreover, the map $\dot{x}\mapsto\mu_{\dot{x}}(A)$ is λ-measurable, and we have

$$\mu(A)=\int_{G/H}\mu_{\dot{x}}(A)d\lambda(\dot{x}),$$

whence the $\mu_{\dot{x}}$ are seen to satisfy our definition of a strict disintegration. ■

Section 2 presents the adjustment and projection theorems together with some immediate consequences of the former. Section 3 is devoted to the strong lifting property; it includes our characterization of this property and the elaboration of Losert's example. Section 4 is devoted to measurable cross sections; specifically, we shall produce (in Theorem 4.4) a λ-measurable right inverse for the map ψ of Example 1.14 which maps compact sets into relatively compact images.

Except in Section 4, our main results all assume the existence of a lifting. They therefore depend for their applicability upon the lifting theorem in its most general form, which, when the measure space (X, Σ, μ) is complete, asserts the existence of a lifting provided that X constitutes the direct sum mod μ of a family of sets of finite measure. (One amalgamates liftings on subsets of finite measure in the manner of Theorem 2.4 below; cf. [3, theorem 1, p. 206].) To a lesser extent they also depend upon the extendability of a lifting from a smaller σ-algebra to a larger one. This fact follows implicitly from the constructive proofs of the lifting theorem (such as in [28]), but appears to be generally valid only when the underyling measure is finite.

2. The adjustment and projection theorems

The results of this section are unified by the general theme: new liftings from old. Our first principal result below will be called the *adjustment theorem.* Essentially it asserts that a lifting may be "adjusted" on a fixed null set in such a way that it becomes strong on that null set with respect to a given class of "open" sets.

2.1. THEOREM. *Let (X, Σ, μ) be a complete measure space, let N be a μ-null set, let $F = N^c$, and let ρ be a lifting for μ on (X, Σ). Let $\mathscr{G}$ be a collection of nonnull subsets of X such that $\mathscr{G} \cup \{\varnothing\}$ is closed under the formation of finite intersections. Then there exists a lifting ρ_N for μ on (X, Σ) such that:*

(2.1.1) *for all $x \in N$, for all $G \in \mathscr{G}$, and for all $A \in \Sigma$, the condition $x \in G \subseteq_\mu A$ implies that $x \in \rho_N(A)$;*

(2.1.2) *if $A \in \Sigma$, then $\rho_N(A) \cap F = \rho(A) \cap F$;*

(2.1.3) *hence, if $G, G^c \in \mathscr{G} \cap \mathrm{ran}(\rho)$, then $\rho_N(G) = \rho(G)$.*

REMARKS. It is important to note that, with obvious changes of notation, this result and its proof remain valid when ρ is a lifting for $\mu|_F$ on $(F, \Sigma|_F)$. Then with $\mathscr{G} = \{X\}$, it gives an "extension" of ρ from $\Sigma|_F$ to Σ (cf. [16, proposition 12,

p. 128]). We do *not* assume that $\mathcal{G} \subseteq \Sigma$. Provided the elements of $\mathcal{G}$ are not μ-null sets, the theorem applies.

PROOF. If $X \notin \mathcal{G}$, let it be added to $\mathcal{G}$. Define a density ρ_N^* for μ on (X, Σ) by

$$\rho_N^*(A) = \sigma^*(A) \cup (\rho(A) \cap F),$$

where $\sigma^*(A)$ denotes the set of points $x \in N$ for which there exists a set $G \in \mathcal{G}$ such that $x \in G \subseteq_\mu A$. The properties of a density are straightforward to verify. For example, let $x \in \rho_N^*(A) \cap \rho_N^*(B)$. If $x \in F$, then necessarily

$$x \in \rho(A) \cap \rho(B) \cap F = \rho(A \cap B) \cap F \subseteq \rho_N^*(A \cap B).$$

If $x \in N$, then there exist sets $G, H \in \mathcal{G}$ such that $x \in G \subseteq_\mu A$, and such that $x \in H \subseteq_\mu B$. It follows that $G \cap H \neq \emptyset$, whence $G \cap H \in \mathcal{G}$, whence $x \in \sigma^*(A \cap B) \subseteq \rho_N^*(A \cap B)$. Note that property (1.6) requires the elements of $\mathcal{G}$ to be nonnull, and it also requires at least that $\bigcup \mathcal{G} = X$.

By the theorem of von Neumann [23], which is given a strikingly simple proof by Traynor in [28, theorem 3, p. 268], we conclude that there is a lifting ρ_N for μ on (X, Σ) such that $\rho_N^*(A) \subseteq \rho_N(A)$ for all $A \in \Sigma$. (The assumption of completeness is used here.) Property (2.1.1) is immediate from the definition of ρ_N^*; property (2.1.3) follows from the relations $\rho(G) = \rho_N^*(G) \subseteq \rho_N(G)$, together with $\rho(G)^c = \rho(G^c) = \rho_N^*(G^c) \subseteq \rho_N(G^c) = \rho_N(G)^c$; and property (2.1.2) is established in similar spirit. ■

Our first application of the adjustment theorem amalgamates and generalizes a number of known results, some of whose original proofs were lengthy.

2.2. THEOREM. *Let (X, Σ, μ) be a complete measure space, and let ρ be a lifting for μ on (X, Σ). Let $\mathcal{S} \subseteq \Sigma$ be a collection of nonnull subsets of X such that $\mathcal{S} \cup \{\emptyset\}$ is closed under the formation of finite intersections, and such that $\mathcal{S}$ is "second countable" in the sense that there exists a sequence $\{B_n\}_{n=1}^\infty$ in Σ with the following property: If $x \in U \in \mathcal{S}$, then there is an index n such that $x \in B_n \subseteq_\mu U$. Let $\mathcal{A}$ be a subalgebra of Σ such that, whenever $U \in \mathcal{S}$ and $A \in \mathcal{A}$ satisfy $\mu(U \cap A) = 0$, we have $\mu(A) = 0$. Then there exists a lifting σ for μ on (X, Σ) such that*:

(2.2.1) $$\sigma(U) \supseteq U \quad \text{for all } U \in \mathcal{S};$$

(2.2.2) $$\sigma(A) = \rho(A) \quad \text{for all } A \in \mathcal{A}.$$

NOTE. $\mathcal{A}$ need *not* be a σ-algebra.

PROOF. If $X \notin \mathcal{S}$, let it be added to $\mathcal{S}$. Apply the adjustment theorem with

$$N = \bigcup_{n=1}^{\infty} (B_n \setminus \rho(B_n)) \quad \text{and} \quad \mathcal{G} = \{U \cap \rho(A) : U \in \mathcal{S};\ A \in \mathcal{A};\ \mu(A) > 0\}.$$

Set $\sigma = \rho_N$. Then, in view of the "second countable" condition, and of the fact that we have $\rho(A) \subseteq \rho(B)$ whenever $A \subseteq_\mu B$, property (2.2.1) follows easily from (2.1.1) and (2.1.2). And property (2.2.2) follows from (2.1.3). ■

When $\mathcal{A} = \{\emptyset, X\}$, Theorem 2.2 (always in conjunction with the standard lifting theorem) reduces to the principal result of Graf in [9]. (This in turn generalizes, for example, [3, proposition 5, p. 407] and [5].) Theorem 2.2 also absorbs the essential content of [16, proposition 3, p. 110; proposition 8, p. 114; theorem 4, p. 115] and, combined with the constructive techniques of [28], will yield (a generalization of) [16, theorem 5, p. 118] with no additional technicalities (cf. [22]).

Our second application of the adjustment theorem is concerned with the "localization" and "globalization" of strong liftings. The following lemma is a preliminary localization result.

2.3. LEMMA. *Let $(X, \Sigma, \mu, \mathcal{T})$ be a topological measure space, let σ be a strong lifting for μ on (X, Σ), and let $E, F \in \Sigma$ be such that $E \approx_\mu F \subseteq \sigma(E)$. Then F is a carrier of μ, and, for $A \in \Sigma|_F$, the identity $\rho(A) = \sigma(A) \cap F$ determines a lifting ρ for $\mu|_F$ on $(F, \Sigma|_F)$ which is strong with respect to the relative topology $\mathcal{T}|_F$.*

PROOF. It is easy to verify that ρ is a lifting. (Note that the condition $F \subseteq \sigma(E)$ is needed to establish property (1.6).) The fact that ρ is strong follows from the relations

$$\begin{aligned} \rho(U \cap F) &= \sigma(U \cap F) \cap F = \sigma(U) \cap \sigma(F) \cap F \\ &= \sigma(U) \cap \sigma(E) \cap F = \sigma(U) \cap F \supseteq U \cap F, \end{aligned}$$

which are valid for all open sets U. It necessarily follows that μ is carried on F. ■

2.4. THEOREM. *Let $(X, \Sigma, \mu, \mathcal{T})$ be a complete topological measure space, and assume that μ is carried on X.*

(2.4.1) *If μ has the strong lifting property, and if E is a carrier of μ, then the space $(E, \Sigma|_E, \mu|_E, \mathcal{T}|_E)$ has the strong lifting property.*

(2.4.2) *If X is the direct sum* mod μ *of a family $\{X_\alpha\}$ of carriers of μ such that $\mu|_{X_\alpha}$ has the strong lifting property (with respect to $\mathcal{T}|_{X_\alpha}$) for all α, then μ itself has the strong lifting property.*

PROOF. To prove (2.4.1), let $F = E \cap \sigma(E)$, and let ρ be the strong lifting for μ on $(F, \Sigma|_F)$ which was defined in Lemma 2.3. Since μ is actually carried on E, we may apply (the alternate version of) the adjustment theorem with $N = E \setminus F$ and $\mathscr{G} = \mathscr{T}|_E$ to "extend" ρ to a strong lifting for $\mu|_E$ on $(E, \Sigma|_E)$.

To prove (2.4.2), let $F = \bigcup_\alpha X_\alpha$. Then by (1.1), we have $F \in \Sigma$, and by (1.2), we have $\mu(F^c) = 0$. For each index α, let σ_α be a strong lifting for $\mu|_{X_\alpha}$ on $(X_\alpha, \Sigma|_{X_\alpha})$. Then, for $A \in \Sigma|_F$, the usual formula

$$\sigma(A) = \bigcup_\alpha \sigma_\alpha(A \cap X_\alpha)$$

defines a strong lifting σ for $\mu|_F$ on $(F, \Sigma|_F)$. Note that property (1.7) requires the X_α to be pairwise strictly disjoint. Since μ is carried on X, we may apply the adjustment theorem (as before) with $N = F^c$ and $\mathscr{G} = \mathscr{T}$. ■

Theorems 2.2 and 2.4 combine with the lifting theorem to yield a general sufficient condition for the existence of a strong lifting. The key conditions are that the space be locally second countable and globally decomposable into a direct sum.

2.5. COROLLARY. *Let $(X, \Sigma, \mu, \mathscr{T})$ be a complete topological measure space. Assume that μ is carried on X, and that X constitutes the direct sum* mod μ *of a (pairwise disjoint) family $\{X_\alpha\} \subseteq \Sigma$ such that, for all α:*

(2.5.1) $$\mu(X_\alpha) < \infty;$$

(2.5.2) *X_α is a carrier of μ;*

(2.5.3) *X_α is second countable in its relative topology.*

Then $(X, \Sigma, \mu, \mathscr{T})$ has the strong lifting property. (Indeed it suffices merely to assume, in place of (2.5.1), *that a lifting for $\mu|_{X_\alpha}$ exists.)* ■

By the arguments of [3, proposition 41, p. 337], a locally compact Hausdorff space equipped with a regular Borel measure μ constitutes the direct sum mod μ of a family of compact carriers of μ. Therefore, in view of Corollary 2.5, Theorem 2.4 generalizes [16, proposition 2, p. 108; theorem 3, p. 109; remark 1, p. 127] as well as [16, theorem 10, p. 131].

2.6. COROLLARY. *Let $(X, \Sigma, \mu, \mathscr{T})$ be a totally σ-finite topological measure space, and let ν be a totally σ-finite measure with domain Σ. Assume that both μ and ν are carried on X. Then the following assertions are valid if it is understood that Σ is first completed with respect to any measure which is asserted to have the strong lifting property:*

(2.6.1) *If μ has the strong lifting property, and if ν is absolutely continuous with respect to μ, then ν has the strong lifting property.*

(2.6.2) *The measure $\mu + \nu$ has the strong lifting property if and only if μ and ν both have the strong lifting property.*

PROOF. We may assume that μ and ν are finite without affecting the respective σ-ideals of null sets. To prove (2.6.1), let X_1 be the complement of a ν-null set of maximal μ-measure. Note that μ and ν share the same null subsets of X_1. If necessary, remove a null set from X_1 so as to ensure that $X_1 \subseteq \sigma(X_1)$, where σ is a strong lifting for μ on (X, Σ). Apply Lemma 2.3 and then (2.4.2) with X the (trivial) direct sum mod ν of the singleton $\{X_1\}$.

Condition (2.6.2) is established in similar spirit. Let $\{X_1, X_2, X_3\}$ be a partition of X such that $\mu(X_1) = 0$; such that $\nu(X_2) = 0$; and such that μ, ν, and $\mu + \nu$ share the same null subsets of X_3. Let σ and τ be strong liftings for μ and ν, respectively, on (X, Σ). If necessary, remove a $(\mu + \nu)$-null set from each X_i so as to ensure that $X_1 \subseteq \tau(X_1)$, that $X_2 \subseteq \sigma(X_2)$, and that $X_3 \subseteq \sigma(X_3) \cap \tau(X_3)$. Apply Lemma 2.3 and then (2.4.2) with X the direct sum mod $\mu + \nu$ of $\{X_1, X_2, X_3\}$. ∎

Our second principal result below will be called the *projection theorem.* Essentially it asserts that, in the presence of a strict disintegration, a strong lifting for the disintegrated measure may be "projected" onto a strong lifting for the disintegrating measure (cf. [12]).

2.7. THEOREM. *Let $(X, \Sigma, \mu, \mathcal{T}_1)$ and $(Y, \Upsilon, \nu, \mathcal{T}_2)$ be topological measure spaces, with the latter space assumed to be complete. Let $\psi : X \to Y$ be a continuous surjection which induces a strict disintegration $\{\mu_y\}_{y \in Y}$ of μ with respect to ν on (X, Σ). Assume that ψ is measurable in the sense that $\psi^{-1}(A) \in \Sigma_\mu$ for all $A \in \Upsilon$. If under these conditions there exists a strong lifting σ for μ on (X, Σ_μ), then ν has the strong lifting property.*

REMARKS. The slight delicacy here is that strict disintegrations, as we have defined them, do not commonly exist on (X, Σ_μ), whereas strong liftings do not commonly exist on (X, Σ). The measurability assumption on ψ is superfluous when ν is finite, for then a strong lifting for ν which is defined on the completion (with respect to ν) of the σ-algebra generated by $\mathcal{T}_2$ can be extended to Υ. The fact that the disintegration must be induced on all of (X, Σ) (rather than on $(X, \mathcal{S})$ for some $\mathcal{S} \subseteq \Sigma$) will necessitate a local application of this theorem in Proposition 4.3 below.

PROOF. If (1.10) or (1.11) fails for any $y \in Y$, replace μ_y by a point mass at an arbitrarily selected point $x \in \psi^{-1}(\{y\})$. By this device, (1.10) and (1.11) become valid for every point $y \in Y$, while (1.12) suffers no damage.

By the von Neumann–Traynor theorem which was used in the proof of Theorem 2.1, it suffices to define a strong density for ν on (Y, Y). To this end, let $A \in \mathrm{Y}$, and define

$$\rho^*(A) = \{y \in Y : \sigma(\psi^{-1}(A)) \approx_{\mu_y} X\}.$$

Let us verify first that $\rho^*(A) \approx_\nu A$. Define $E = \psi^{-1}(A) \triangle \sigma(\psi^{-1}(A))$. Then we have $E \subseteq N$, where $N \in \Sigma$, and $\mu(N) = \int \mu_y(N) d\nu(y) = 0$. Therefore there exists a "bad" ν-null set B such that E is μ_y-null for all $y \notin B$. We propose to show that $\rho^*(A) \triangle A \subseteq B$. This we establish by fixing a point $y \in B^c$ and by showing that $y \in A$ if and only if $y \in \rho^*(A)$. The significance of $y \in B^c$ is that $\psi^{-1}(A) \approx_{\mu_y} \sigma(\psi^{-1}(A))$. If $y \in A$ as well, then we have $\psi^{-1}(A) \approx_{\mu_y} X$ simply by the fact that μ_y is supported on $\psi^{-1}(\{y\}) \subseteq \psi^{-1}(A)$. Hence $y \in \rho^*(A)$. Conversely, if $y \in \rho^*(A)$ (as well as B^c), then we have $X \approx_{\mu_y} \sigma(\psi^{-1}(A)) \approx_{\mu_y} \psi^{-1}(A)$. But $\mu_y(X) > 0$, whence $y \in A$, for otherwise we achieve the contradiction $\psi^{-1}(A) \approx_{\mu_y} \varnothing$.

The remaining properties of a density are straightforward to verify. For example, let $y \in \rho^*(A) \cap \rho^*(B)$, where now the sets $A, B \in \mathrm{Y}$ are arbitrary. Then we have $\sigma(\psi^{-1}(A)) \approx_{\mu_y} X \approx_{\mu_y} \sigma(\psi^{-1}(B))$, so that $X \approx_{\mu_y} \sigma(\psi^{-1}(A)) \cap \sigma(\psi^{-1}(B)) = \sigma(\psi^{-1}(A \cap B))$, whence $y \in \rho^*(A \cap B)$. Note that property (1.6) (also) requires that $\mu_y(X) > 0$ for *all* $y \in Y$.

It remains to show that $\rho^*(U) \supseteq U$ for all $U \in \mathcal{T}_2$. By continuity of ψ, and by the fact that σ is strong, we have, for all $y \in U$, that $\psi^{-1}(\{y\}) \subseteq \psi^{-1}(U) \subseteq \sigma(\psi^{-1}(U))$, so that, again by (1.11), $\sigma(\psi^{-1}(U)) \approx_{\mu_y} X$, giving $y \in \rho^*(U)$. ∎

3. The existence and non-existence of strong liftings

This section is devoted to a counterexample and a theorem concerning the existence of strong liftings. Losert's celebrated counterexample [20] (cf. [8]) settled the long-standing question as to whether a regular Borel measure carried on a compact Hausdorff space has the strong lifting property. A negative example of a similar sort appears in [27, theorem 5, p. 171]. We shall employ the projection theorem to extend Losert's example to compact Hausdorff spaces of arbitrarily high weight, where the *weight* of any topological space is defined to be the smallest cardinal number κ for which there exists a neighborhood basis for the topology of cardinality κ.

To avoid trivialities we introduce the following concept.

3.1. Definition. It will be said that a topological measure space $(X, \Sigma, \mu, \mathcal{T})$ (or just μ) *everywhere fails the strong lifting property* if, for all $A \in \Sigma$, the space $(A, \Sigma|_A, \mu|_A, \mathcal{T}|_A)$ fails to have the strong lifting property. ■

It is not (now) difficult to see that such spaces exist. If Losert's example does not already possess this property, then, in view of Theorem 2.4, a standard exhaustion argument will yield a subspace which does. And by the regularity of the measure, the subspace can be taken to be compact.

It is perhaps not surprising that an example of this sort will "poison" any product of which it constitutes a factor.

3.2. Example. For each cardinal number $\kappa \geqq \aleph_2$, we shall exhibit a regular Borel measure carried on a compact Hausdorff space of weight κ which everywhere fails the strong lifting property. The foregoing discussion yields a compact Hausdorff space X of weight $\leqq \aleph_2$ and a regular Borel measure μ carried on X which everywhere fails the strong lifting property. Let ν be any regular Borel measure which is carried on a compact Hausdorff space Y of weight κ. (For example, ν might be product Lebesgue measure on κ many copies of the unit interval $[0,1]$.) Then $X \times Y$ has weight κ, and we propose to show that the measure $\mu \times \nu$, regarded as a regular Borel measure on $X \times Y$, everywhere fails the strong lifting property.

To this end, first recall the notation and content of Example 1.13. Now suppose, to the contrary, that $\mu \times \nu|_{C_0}$ does possess that strong lifting property (always with respect to the relative topology) for some set $C_0 \in \Sigma$. By Theorem 2.4, C_0 may be supposed to be compact. Define $K_0 = \pi(C_0)$, and let K be any nonnull compact carrier of μ such that $K \subseteq \{x \in X : \nu_x(C_0) > 0\}$. Now define $C = C_0 \cap \pi^{-1}(K)$. Then by property (1.11), we have $\nu_x(C) > 0$ for every point $x \in K$. If necessary, remove a relatively open $(\mu \times \nu)$-null set from C so as to ensure that C is a carrier of $\mu \times \nu$. Since K is a carrier of μ, the function π will still map C onto K, whereas the function $\nu_x(C)$ will become 0 for at most a μ-null set of points x in K. It follows that the family $\{\nu_x|_C\}_{x \in K}$ constitutes a strict disintegration of $\mu \times \nu|_C$ with respect to $\mu|_K$ induced on $(C, \mathcal{B}(C))$ by $\pi|_C$. By Theorem 2.4, $\mu \times \nu|_C$ has the strong lifting property, so that, by the projection theorem, $\mu|_K$ will inherit this property. But then the fact that μ everywhere fails the strong lifting property is contradicted. ■

This argument shows, in particular, that if $\mu \times \nu$ does have the strong lifting property, then so also do μ and ν.

3.3. Question. Is the converse true? If μ and ν are regular Borel measures on compact Hausdorff spaces, and if μ and ν both have the strong lifting property, then does the measure $\mu \times \nu$ have the strong lifting property? ■

These examples all highlight the importance of characterizing the strong lifting property. The literature contains at least two such characterizations, one [12] in terms of the existence of strict disintegrations, and another [13] in terms of the existence of measurable right inverses (see also [21, theorem 1, p. 154]). We now propose an internal characterization of strong liftings in terms of their behavior on the topology $\mathcal{T}$.

If σ is a strong lifting for a measure μ, and if U is an open set, then the set $\sigma(U)$ constitutes a "measure theoretic closure" of U which satisfies (at least) properties (3.4.1)–(3.4.6) below. With mild restrictions upon μ, the existence of a "closure" operation which satisfies these properties will be sufficient to guarantee the existence of a strong lifting for μ. The original idea of the proof was to adjust an arbitrary lifting ρ at every (null) point x so as to obtain a lifting ρ_x which was strong at x, and then to piece the ρ_x together into a global strong lifting. However, once it became clear what conditions were needed to implement the piecing together procedure, the ρ and the ρ_x were seen to be superfluous. What remained was a manipulation with ultrafilters which owes a clear debt to the proof of theorem 3 in [28, p. 268], and also to the proof of Theorem 2.1 Here is the result.

3.4. THEOREM. *Let* $(X, \Sigma, \mu, \mathcal{T})$ *be a complete topological measure space. Assume that* μ *is regular (in the sense of* (1.3)) *and that* X *is the direct sum* mod μ *of a family* $\{X_\alpha\}$ *of sets of finite measure. For each open set* U*, let there be given a set* $\mathrm{cl}(U) \in \Sigma$ *with the following properties, which are to hold for all* $U, V \in \mathcal{T}$:

(3.4.1) $$\mathrm{cl}(U) \subseteq \bar{U};$$

(3.4.2) $$\mathrm{cl}(U) \approx_\mu U;$$

(3.4.3) $$\text{if } U \approx_\mu X, \text{ then } \mathrm{cl}(U) = X;$$

(3.4.4) $$\mathrm{cl}(U \cup V) \subseteq \mathrm{cl}(U) \cup \mathrm{cl}(V).$$

Assume, moreover, either *that* X *satisfies the* T_3 *separation axiom,* or *that* cl *satisfies the following two conditions, also valid for all* $U, V \in \mathcal{T}$:

(3.4.5) $$U \subseteq \mathrm{cl}(U);$$

(3.4.6) $$\text{if } U \approx_\mu V, \text{ then } \mathrm{cl}(U) = \mathrm{cl}(V).$$

Then there exists a lifting σ *for* μ *on* (X, Σ) *which satisfies* $U \subseteq \sigma(U) \subseteq \mathrm{cl}(U)$ *for every open set* U.

NOTE. The X_α need *not* be carriers of μ; indeed, μ plays a minimal role in the proof to follow.

PROOF. For $x \in X$, define

$$\mathcal{K}_x = \{K \subseteq X : K \text{ closed; } x \in \mathrm{cl}(K^c)^c\}.$$

Then assumption (3.4.1) implies that $\mathcal{K}_x$ contains the closed neighborhoods of x; assumption (3.4.2) implies that, for every closed set K, we have $K \in \mathcal{K}_x$ for almost all $x \in K$; assumption (3.4.3) implies that $\mathcal{K}_x$ contains no μ-null sets; and assumption (3.4.4) implies that $\mathcal{K}_x$ is closed under the formation of finite intersections. Therefore, by Zorn's lemma, there exists an ultrafilter $\mathcal{U}_x$ which contains $\mathcal{K}_x$.

Let us assume for the moment that the $\mathcal{U}_x$ may be obtained in such a way that

(3.4.7) for all $x \in X$, for all $U \in \mathcal{T}$, and for all $A \in \Sigma$, the condition $x \in U \subsetneq_\mu A$ implies that $A \in \mathcal{U}_x$.

Then for $A \in \Sigma$, define

$$\sigma(A) = \{x \in X : A \in \mathcal{U}_x\}.$$

We propose to show that σ is the desired lifting. Properties (1.6)–(1.8) are immediate from the properties of ultrafilters (cf. [28, p. 268]). In view of this, property (1.5) will follow once it is seen that no null set N can belong to any of the $\mathcal{U}_x$ (from which $\sigma(N) = \varnothing$ follows). However, since $X \subsetneq_\mu N^c$, this is immediate from condition (3.4.7). It is property (1.4) whose proof will utilize the stated assumptions about the measure μ. By (1.6)–(1.8), it suffices to prove that almost every point of an arbitrary set $A \in \Sigma$ belongs to $\sigma(A)$. If $\mu(A) < \infty$, then, by the regularity of μ, there is a sequence $\{K_n\}$ of closed subsets of A such that $A \approx_\mu \bigcup_{n=1}^{\infty} K_n$. Since $K_n \in \mathcal{K}_x \subseteq \mathcal{U}_x$ for almost every $x \in K_n$ (and for all n), it follows that almost every x in each of the K_n, and hence almost every x in A, belongs to $\sigma(A)$. If $\mu(A) = \infty$, the foregoing argument applies to each of the sets $A \cap X_\alpha$, so that (1.2) yields the same conclusion. This establishes (1.4), and hence the fact that σ is a lifting for μ on (X, Σ). That σ is strong follows at once from condition (3.4.7); that $\sigma(U) \subseteq \mathrm{cl}(U)$ for all $U \in \mathcal{T}$ follows by observing that if $x \in \mathrm{cl}(U)^c$, then $U^c \in \mathcal{K}_x \subseteq \mathcal{U}_x$, so that $x \in \sigma(U^c) = \sigma(U)^c$.

To complete that proof of the theorem, it remains to verify that the $\mathcal{U}_x$ may be chosen in such a way as to satisfy condition (3.4.7). To this end it suffices to show that any set A as in (3.4.7) has nonnull, and hence nonempty, intersection with an arbitrarily chosen set $K \in \mathcal{K}_x$; moreover, it suffices to assume as well that A is an open neighborhood U of x. If X is a T_3 space, then, by definition, U

contains a closed neighborhood W of x. In this case we already know that $W \cap K$, and hence $U \cap K$, are nonnull. Let us deduce the same conclusion from (3.4.5) and (3.4.6) by deriving a contradiction if it is supposed that $\mu(U \cap K) = 0$. For then we would have $K^c \cup U \approx_\mu K^c$, so that $x \in U \subseteq K^c \cup U \subseteq \mathrm{cl}(K^c \cup U) = \mathrm{cl}(K^c)$. But $K \in \mathscr{K}_x$, so that, by definition of $\mathscr{K}_x$, we have $x \notin \mathrm{cl}(K^c)$. ∎

3.5. Discussion. It is to be emphasized that the "cl" operation must be defined for *every* open set, and not, say, for just a collection of basic open sets. For example, if we restrict our attention to the open sets with null boundary, then the ordinary closure operation, $\mathrm{cl}(U) = \bar{U}$, will clearly satisfy (3.4.1)–(3.4.6). If μ is a Borel measure on a locally compact Hausdorff space, then, by Urysohn's lemma and the little argument in [4, p. 316], these sets form a basis for the topology; nevertheless, by Example 3.2, μ can everywhere fail the strong lifting property. We therefore do not anticipate any trivial applications of Theorem 3.4.

A lack of trivial applications may suggest that, from the measure theoretic point of view, an arbitrary open set is scarcely less mysterious than an arbitrary measurable set. For example, it is easy enough to see that there exists a (strong) lifting σ for Lebesgue measure such that $\sigma[a, b) = [a, b)$ whenever $a < b$. Thus: if A is a Lebesgue measurable subset of the real line, define $F(x) = \int_0^x \chi_A(t)dt$, and apply the classical Lebesgue theory to deduce that the derivative F' is well defined and equal to χ_A almost everywhere. It follows that the identity

$$\sigma^*(A) = \left\{ x : \lim_{h \to 0^+} \frac{F(x+h) - F(x)}{h} = 1 \right\}$$

determines a density σ^* for which any corresponding lifting σ (i.e. such that $\sigma^*(A) \subseteq \sigma(A)$) is clearly as desired (cf. [16, theorem 6, p. 123]). The point is that if the classical theory were any easier to establish for an arbitrary closed (as opposed to measurable) set, then, with $\mathrm{cl}(U) = \sigma^*(U^c)^c$, Theorem 3.4 would yield a corresponding simplification in the construction of σ. ∎

4. Measurable cross sections in locally compact groups

As pointed out in [6], many constructions in the theory of locally compact groups depend for their validity upon the existence of "reasonably well behaved" cross sections for the cosets G/H (as defined in Example 1.14). The last decade has witnessed a burgeoning of general measurable selection theorems (see [29] and [30]), and it is the aim of this section to demonstrate that one of the most general of these is applicable in the group setting.

We shall use the notation and content of Example 1.14 throughout, except that an arbitrary element of G/H will often now be denoted by y in place of $\dot{x}$. Also define $\Sigma = \mathcal{B}_l(G)_\mu$, and define $\Upsilon = \mathcal{B}_l(G/H)_\lambda$. To facilitate the application of the projection theorem, we exploit the existence of a nonnegative, bounded, continuous function $\mathfrak{b}$ on G such that

$$\int_H \mathfrak{b}(xs)d\beta(s) = 1$$

for all $x \in G$ ([2, p. 103]; see also [2, proposition 8, p. 51]). Let $S_\mathfrak{b}$ denote the support of $\mathfrak{b}$, i.e. the closure of the set $\{x \in G : \mathfrak{b}(x) > 0\}$. Then we may, and shall, assume that $S_\mathfrak{b}$ has compact intersection with the saturant CH of every compact subset of C of G. Since the canonical map ψ is open as well as continuous, it follows that

(4.1) $$\text{for every compact set } K \subseteq G/H,$$
$$\text{the set } \psi^{-1}(K) \cap S_\mathfrak{b} \text{ is compact.}$$

Another vital property of the set $S_\mathfrak{b}$ is the obvious fact that $\int_H \chi_{S_\mathfrak{b}}(xs)d\beta(s) > 0$ for all $x \in G$. Since the quotient Δ/δ is strictly positive on H, it follows that we have

(4.2) $$\mu_{\dot{x}}(S_\mathfrak{b}) = \rho(x)^{-1} \int_H \frac{\Delta(s)}{\delta(s)} \chi_{S_\mathfrak{b}}(xs)d\beta(s) > 0$$

for all $x \in G$.

The existence of measurable selections in the absence of separability assumptions *almost* requires a strong lifting [21, theorem 1, p. 154], and indeed the precise measurability properties of the selection can be related to corresponding properties of the Boolean algebra homomorphism which is used to construct it [21, remark, p. 155]. Therefore the following application of the projection theorem appears to be crucial.

4.3. PROPOSITION. *The quasi-invariant measure λ on G/H has the strong lifting property.*

PROOF. It is obvious that λ is carried on G/H. By Theorem 2.4 and the arguments in [3, proposition 41, p. 337], it suffices to obtain a strong lifting for $\lambda|_K$ on $(K, \Upsilon|_K)$, where K is a compact carrier of λ. To this end, define $C = \psi^{-1}(K) \cap S_\mathfrak{b}$, and note that C is compact by (4.1). Upon removing a μ-null set from C, if necessary, we may ensure the existence of a strong lifting for $\mu|_C$ on $(C, \Sigma|_C)$ by applying Theorem 2.4 and the vital [15, proposition 1, p. 66]. In view of (1.11) and (4.2) the fact that this lifting may be projected onto a strong lifting for $\lambda\,|_K$ follows by the arguments of Example 3.2 with μ, λ, ψ, and μ_y $(y = \dot{x})$ in place of $\mu \times \nu$, μ, π, and ν_x, respectively. ■

We are now prepared for the major result of this section, which asserts the existence of a λ-measurable cross section from G/H into G.

4.4. THEOREM. *Let the notation of Example* 1.14 *be given. Then there exists a function* $\theta : G/H \to G$ *with the following properties*:

(4.4.1) $$\text{for all } y \in G/H, \text{ we have } \psi(\theta(y)) = y;$$

(4.4.2) $$\text{for all } A \in \mathscr{B}_\delta(G) \,(\text{resp. } \mathscr{B}_\sigma(G); \text{ resp. } \mathscr{B}_1(G)),$$
$$\text{we have } \theta^{-1}(A) \in \mathscr{B}_\delta(G/H)_\lambda$$
$$(\text{resp. } \mathscr{B}_\sigma(G/H)_\lambda\,; \text{ resp. } \mathscr{B}_l(G/H)_\lambda = \mathrm{Y});$$

(4.4.3) $$\text{for every compact set } K \subseteq G/H,$$
$$\text{the set } \theta(K) \text{ is relatively compact in } G.$$

REMARKS. In fact we shall obtain such a θ with values in S_b. The key selection theorem of Graf [10, theorem 5, p. 348] may be applied either locally or globally. The former approach seems to be marginally easier; moreover, the special case of Graf's theorem which is then utilized admits a relatively simple proof.

PROOF. Let K be a compact carrier of λ, and define the compact set $C = \psi^{-1}(K) \cap S_b$ as before. We propose to apply Graf's theorem with $(K, \mathrm{Y}\,|_K, \lambda\,|_K)$ in the role of Graf's space $(X, \mathscr{A}, \mu)$, and with C in the role of Graf's space Y. It will yield a right inverse θ for $\psi\,|_C$ which is measurable with respect to $\mathrm{Y}\,|_K$ and $\mathscr{B}(C)$, where $\mathscr{B}(C)$ denotes the usual Borel subsets of C [10, p. 346]. To this end, for $y \in Y$, define $F(y)$ to be the nonempty compact set $\psi^{-1}(\{y\}) \cap C$. We have seen in Proposition 4.3 that $\lambda\,|_K$ has the strong lifting property, so it remains to verify that the set function F is upper semicontinuous in the sense of [10, p. 347]. But this is immediate, for if $A \subseteq C$ is closed, then the set $F^{-1}(A)$, as defined in [10, p. 343], is simply the compact set $\psi(A)$. The applicability of Graf's theorem follows.

To complete the proof of the theorem, we globalize this result. Following again the arguments of [3, proposition 41, p. 337], we realize G/H as the direct sum mod λ of a family $\{K_\alpha\}$ of compact carriers of λ. With C_α defined for each α to be the set $\psi^{-1}(K_\alpha) \cap S_b$, we determine θ on K_α, with values in C_α, exactly as in the last paragraph. If $y \notin \bigcup_\alpha K_\alpha$, let $\theta(y)$ be an arbitrarily chosen element of

$\psi^{-1}(\{y\}) \cap S_b$. Then θ, thus defined, satisfies condition (4.4.1) by construction, and, in view of (4.1), satisfies condition (4.4.3) by virtue of being S_b-valued. Since $\mathcal{B}(C_\alpha) = \mathcal{B}_l(G)|_{C_\alpha}$ for each α, it is immediate from property (1.1) that $\theta^{-1}(A) \in \Upsilon$ whenever $A \in \mathcal{B}_l(G)$. The measurability of θ with respect to $\mathcal{B}_\sigma(G/H)_\lambda$ and $\mathcal{B}_\sigma(G)$ follows once it is recognized: first, that every set $A \in \mathcal{B}_\sigma(G)$ is contained in a countable union of compact sets (so that the inverse image $\theta^{-1}(A) \subseteq \psi(A)$ has the same property); and second, that we have $\mathcal{B}_\sigma(G/H)_\lambda|_B = \Upsilon|_B$ for every set $B \in \mathcal{B}_\sigma(G/H)$. A similar argument shows the measurability of θ with respect to $\mathcal{B}_\delta(G/H)_\lambda$ and $\mathcal{B}_\delta(G)$. Thus condition (4.4.2) is established, and the proof is complete. ∎

The most usual employment of cross sections in the group setting involves their combination with the group (or other) operations into more elaborate functions f, such as $f(x) = \theta(\dot{x})^{-1}x$ ($x \in G$). (See [25, p. 872], [11, p. 92], [18, p. 167; lemma 1, p. 168], and [19, p. 66].) These functions are meant to inherit measurability properties from corresponding properties of θ. If θ satisfies an "inverse image" criterion of measurability, as in Theorem 4.4, then in general, because of the delicacies involved in product σ-ring structure, it only seems possible to establish that the f inverse images of *Baire* sets are measurable, and measurable moreover in the same sense that the θ inverse images of locally Baire sets are measurable. (For the particular f and θ above, the f inverse image of a Baire subset of H will belong to $\mathcal{B}_l(G)_\mu$.) This state of affairs seems to be adequate for the applications of [18] and [11], but not for [25], and so, in view of [21, remark, p. 155], we ask:

4.5. Question. Can the cross section θ be constructed to be Borel measurable (in the sense of [6, theorem 1, p. 456]), or Baire measurable (in the sense of [25, proposition 1, p. 872])? ∎

The somewhat delicate measurability questions associated with the inverse image criteria of measurability become considerably more transparent if θ can be taken to be *Lusin λ-measurable*. By this we mean: For every set E of finite λ-measure, and for every number $\varepsilon > 0$, there exists a compact subset K of E such that $\lambda(E \setminus K) < \varepsilon$, and such that $\theta|_K$ is continuous. (For example, if θ is Lusin λ-measurable, then the particular function f above is easily seen to be Lusin μ-measurable, and hence measurable with respect to $\mathcal{B}_l(G)_\mu$ and $\mathcal{B}_\sigma(H)$.) Since this condition is explicitly assumed in [11, p. 89], we ask:

4.6. Question. Is the cross section θ which was obtained in Theorem 4.4 Lusin λ-measurable? ∎

From properties (4.4.2) and (4.4.3) alone, the answer is certainly *yes* when the group G is first countable. Beyond this we do not know what the situation is, and so we ask, finally:

4.7. Question. Can strong liftings be used to obtain Lusin measurable selection mappings? ■

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Department of Mathematics
Monash University
Clayton, Victoria 3168, Australia